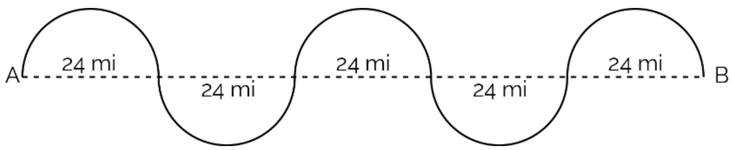
Snakes on a Plain

Teacher Stuff: Solutions with Teaching Tips

CHALLENGE 1

Solution 1 The answer presented here is for AB = 120 mi. Students will have different values for AB.



Divide AB into five congruent segments. Then find the length of one semicircle.

$$\text{Length of one semicircle} = \frac{24\pi}{2} = 37.699 \text{ miles} \quad (1\text{-}1)$$

Multiply by the number of semicircles.

$$\text{Total length of river} = 37.699 \cdot 5 = 188.5 \text{ mi} \quad (1\text{-}2)$$

Solution 2 As students repeat this calculation for different numbers of semicircles, they discover the same answer.

CHALLENGE 2

Solution 4 Students should follow the pattern of the examples they have completed.

Divide AB by the number of semicircles.

$$\text{Length of one segment} = \frac{AB}{n} \quad (2\text{-}1)$$

Find the length of one semicircle.

$$\text{Length of one semicircle} = \frac{\pi AB}{2n} \quad (2\text{-}2)$$

Multiply by the number of semicircles.

$$\text{Total length of } \frac{\pi AB}{2n} \cdot n = \frac{\pi AB}{2} \quad (2\text{-}3)$$

Teaching Tip #1

You'll need to give each student or team of students a different value for AB. This has two advantages. First, it obviously keeps students from just sharing answers. More importantly, later when you do discuss answers as a class, students will see that the results are independent of any specific river length.

By the way, you can also give different groups of students rivers with numbers of semicircles other than 2 and 10. Some of your students will experiment on their own.

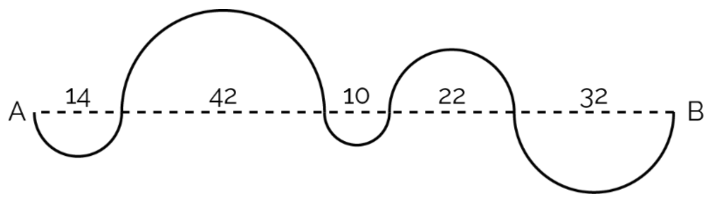
Teaching Tip #2

This step presents a very important teachable moment. When you check in with students before initialing their work, ask them if they noticed what disappeared in the final expression. Then ask them what that says about which variables affect or don't affect the length of the river. Their understanding of this is essential to the rest of the project.

Solution 5 When you initial student work, they should be able to tell you that they notice that n is not in the expression. As they already discovered, the length of the river is independent of the number of semicircles.

CHALLENGE 3

As before, students' rivers and the way they split them up will differ. The procedure is the same.



Solution 7 $14+42+10+22+32=120 \text{ mi}$ (3-1)

Solution 8 Answers will vary, but all students should choose segment lengths that add up to AB.

$$\frac{\pi 14}{2}+\frac{\pi 42}{2}+\frac{\pi 10}{2}+\frac{\pi 22}{2}+\frac{\pi 32}{2}=188.5 \quad (3\text{-}2)$$

Teaching Tip #3

Students are very likely to approach this in a more haphazard fashion than what is shown here. When you check in with them to sign, you can help them see that this is a way to organize the thinking they have already done. This will help them with the next step.

CHALLENGE 4

Students have already done the thinking for this step. (See Teaching Note #3.) Now it's a matter of doing it with variables to represent the diameters.

Solution 10 Add the length of n semicircles.

$$d_1+d_2+d_3+\ldots\ +d_n \quad (4\text{-}1)$$

Replace the sum of the diameters with D.

$$L=\frac{\pi D}{2} \quad (4\text{-}2)$$

Solution 11 Students should be able to tell you that this is the same result that they got from the same three challenges.

Solution 12 They should also be able to reason that this result will occur for any number of segments that add up to AB.

Teaching Tip #4

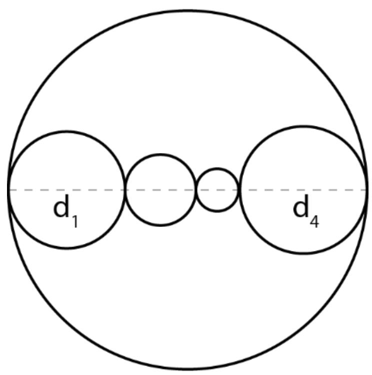
This is the coolest part of the whole project. If your students are like ours, they'll be pretty excited when they make this discovery.

Students will need varying amounts of hinting to get here, but the more you can let them arrive at this result on their own, the stronger they'll be for having done so.

CHALLENGE 5

Solution 14 This problem is structured more loosely on purpose. Students have an opportunity to analyze this new situation using some of the patterns and thinking from the first four challenges. Some students may solve the problem first with congruent circles before moving to an example with circles that have different diameters. Others might be able to move to the highest level of generalization right away.

This generalized solution for four circles is shown here.



Represent the circumference of each circle, and then add them.

Sum of circumferences = $\pi d_1 + \pi d_2 + \pi d_3 + \pi d_4$ (5-1)

Factor out π. Let D = diameter of the big circle.

$\pi\left(d_1 + d_2 + d_3 + d_4\right) = \pi D$ = circumference of big circle. (5-2)

This challenge is a great opportunity to talk about how to develop a clearly written solution.

Teaching Tip #5

Giving students an opportunity to think independently through an analysis helps them develop a skill that employers desperately want. In this case, students have had the advantage of being led first through a similar analysis, but that does not always need to be the case.

Teaching Tip #6

The conversations you'll have with your students as they create differing solutions to this problem are priceless—for you as well as for them. The more you create this type of opportunity for students, the less you'll want to be lecturing from the front of the room.

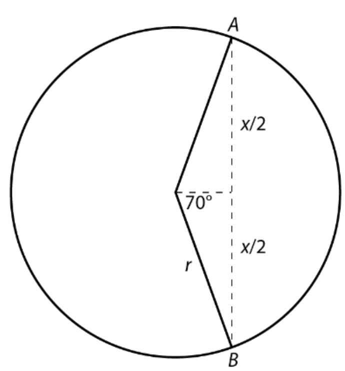
CHALLENGE 6

Solution 15 Let's calculate the length of a single arc, as shown by $\widehat{AB}$.

$$r = \frac{x}{2\sin 70°} = 0.5321x$$

$$\text{Arc length} = \frac{140}{360} \bullet 2\pi r = 1.30x$$

(6.1)



Teaching Tip #7

You can decide whether or not the rest of the challenges are right for your students. The expectation is that students will be more independent in making connections and generalizing and explaining results.

total river length = $3 \bullet \frac{120}{3} \bullet 1.30 = 156 \text{ mi}$ (6-2)

Solution 16 Students should get the same result no matter how many arcs they use. Some students may want to skip this step because they already see that all we have changed is the fraction by which we multiply the circumference. In our opinion, this is okay as long as they can provide a clear explanation that demonstrates that they understand the principle involved.

Solution 17 Here is the generalized result:

$$1.30d_1 + 1.30d_2 + \ldots + 1.30d_n = 1.30\left(d_1 + d_2 + \ldots + d_n\right) \quad (6.2)$$

Finally, $L = 1.30D$, which should seem familiar. (6.3)

Solution 18 The 1.30 that was factored out is directly related to the arc measure, so changing this arc measure for each section of the river will prohibit you from using the distributive property and getting tidy results.

Challenge 7

Solution 20 Let d represent the length of one segment when $\overline{AB}$ is split up. The lengths of the two sides of the triangle in that section are then $\frac{1}{2}d$ and $\frac{\sqrt{3}}{2}d$. Combine these to represent the length of one section of the river for one triangle: $d\left(\frac{1+\sqrt{3}}{2}\right)$. If you add expressions for each of the river sections, you'll get:

$$L = \left(\frac{1+\sqrt{3}}{2}\right)\left(d_1 + d_2 + d_3 + \ldots + d_n\right) = \left(\frac{1+\sqrt{3}}{2}\right)D \quad (7.1)$$

Teaching Tip #8

There are of course other possibilities to explore with specific types of triangles. Maybe some groups can use these triangles while others use isosceles triangles or 3:4:5 triangles. Students have analyzed the basic concepts enough by this point to be able to explore and play.

Solution 21 For all of our examples, the length of one section of the river is found by multiplying the straight-line length of that section by some constant. As long as the river is made up of similar shapes, each section length will be multiplied by this same constant, which can later be factored out of the sum of the sections.

Solution 21 semicircular river: k = 1.57; 140°-arc river: k = 1.22; 30°–60°–90° river: k = 1.37.

CHALLENGE 8

The formula development for this triangle is the same as for the 30-60-90° triangle. The two sides will be $d\sin\theta$ and $d\cos\theta$. One section can be represented by $d(\sin\theta+\cos\theta)$.

Solution 24 The result is $L=(\sin\theta+\cos\theta)D$. (8-1)

CHALLENGE 9

Solution 26 $k=\sin\theta+\cos\theta$ (9-1)

Solution 27 Graphing this function shows that the largest value for *k* turns out to occur when $\theta=45°$. At that angle, *k* = 1.414. (Feel free to find this same result using calculus... just for fun.)

Teaching Tip #9

Are you and your students still hungry for more? Try designing rivers that result in specific values for *k*. How about *k* = 1.5 or *k* = 2? You already know that triangular rivers and rivers based on arcs won't work. Why not? What other shapes could work? There is room for creativity in the math classroom. Have fun!

CHALLENGE 10

It's weird to think that after all this analysis the idea that yields such cool results is simple: similar figures. In similar figures, a given distance will be equal to some other distance multiplied by a constant. Continuing with the variables used throughout this project, we can write:

$$L=kd_1+kd_2+\ldots+kd_n=kD \quad (10.1)$$

That's really all there is to it. It will be interesting to see if any of your students think of this generalization before they get to this last challenge.